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# BRST invariant Lagrangian of spontaneously broken gauge theories in noncommutative geometry

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The quantization of spontaneously broken gauge theories in noncommutative geometry(NCG) has been sought for some time, because quantization is crucial for making the NCG approach a reliable and physically acceptable theory. Lee, Hwang and Ne'eman recently succeeded in realizing the BRST quantization of gauge theories in NCG in the matrix derivative approach proposed by Coquereaux et al. The present author has proposed a characteristic formulation to reconstruct a gauge theory in NCG on the discrete space $M_4 \times Z_N$. Since this formulation is a generalization of the differential geometry on the ordinary manifold to that on the discrete manifold, it is more familiar than other approaches. In this paper, we show that within our formulation we can obtain the BRST invariant Lagrangian in the same way as Lee, Hwang and Ne'eman and apply it to the SU(2)×U(1) gauge theory.

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## I. INTRODUCTION

The effort made to understand the Higgs mechanism has produced many attempts including the Kaluza-Klein model [2], [3], the technicolor model, top quark condensation [4], and the approach based on noncommutative geometry(NCG) on the discrete space. Among these, the NCG approach is most promising because the reason for the existence of the Higgs field becomes apparent, and no extra physical modes are necessary. Since Connes [1] proposed the first idea concerning NCG, many works [2]∼ [15] have appeared, realizing the unified picture of gauge and Higgs fields as the generalized connection on the discrete space $M_4 \times Z_2$.

We have also presented a characteristic formulation [10]∼ [13] which is the generalization of the usual differential geometry on an ordinary manifold to the discrete space $M_4 \times Z_N$. In noncommutative geometry on $M_4 \times Z_2$, the extra differential one-form $\chi$ is introduced in addition to the usual one-form $dx^\mu$, and therefore our formulation is very similar to ordinary differential geometry. In contrast, the original formulation by Connes is very difficult to understand. The one-form basis $\chi$ was originally introduced by Sitarz [8]. However, his scheme is somewhat difficult to apply to the reconstruction of the model in particle physics such as the Standard Model and Grand Unified Model(GUT). We improved Sitarz's scheme by introducing a symmetry breaking matrix to enable us to reconstruct gauge theories such as the Standard Model [21], the left-right symmetric gauge theory [14], SU(5) GUT [11], and SO(10) GUT [13].

Among the many attempts to reconstruct the Standard Model in NCG, the gauge invariant Lagrangian for bosonic and fermionic sectors has been obtained. However, the way to construct the gauge fixing term and ghost term in the NCG scheme to this point has not been evident, though these terms are very important to ensure the quantization of the gauge theory. Until now, these terms have been added by hand, without justification by any reasonable method in NCG. Lee, Hwang and Ne'eman [16] succeeded very recently in obtaining the BRST quantization of a gauge theory in NCG in the matrix derivative approach due to Coquereaux and others [7]. They obtained the BRS/anti-BRS transformation rules of the theory by applying the horizontality condition in the super field formulation and constructed BRST invariant Lagrangian including the gauge fixing and ghost terms. This construction yielded two important features, that the t'Hooft gauge is obtained and that the odd part in matrix formulation produces a global symmetry.

In this article, we apply similar method to our formulation in order to obtain the BRST invariant Lagrangian of the spontaneously broken gauge theory in NCG. We find results similar to those of Lee, Hwang and Ne'eman [16]. The fermion sector is not treated because it is not related to the main theme of this article.

This article consists of four sections. The next section presents the general settings of our formulation by introducing the Grassmann number $\theta$ and $\bar{\theta}$ in addition to $x_\mu$ in $M_4$ and $y$ in the discrete space $Z_2$. Ghost fields are given as members of generalized gauge field in the same way as gauge and Higgs fields. The horizontality condition is applied to obtain the BRS/anti-BRS transformation(BRST/anti-BRST)of the respected fields, and the method to obtain the

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gauge fixing and ghost terms is presented. The third section is an application to the SU(2)×U(1) gauge model, where results very similar to those of [16] are obtained. The last section is devoted to concluding remarks.

## II. GENERAL SETTINGS

In the previous formulation [10], [21], we adopted the discrete space $M_4 \times Z_2$ to reconstruct the Standard Model. According to the super field formulation of Bonora and Tonin [17], the Grassmann numbers $\theta$ and $\bar{\theta}$ are added to $x_\mu$ and $y$ in $M_4 \times Z_2$ to produce the ghost and anti-ghost fields. However, we do not identify the super-space of arguments $\theta$ and $\bar{\theta}$ because this is not necessary to obtain the final results.

Let us start with the equation of the generalized gauge field $\mathcal{A}(x, y, \theta, \bar{\theta})$ written in one-form on the discrete space $M_4 \times Z_2$:

$$\mathcal{A}(x, y, \theta, \bar{\theta}) = \sum_i a_i^\dagger(x, y, \theta, \bar{\theta})\mathbf{d}a_i(x, y, \theta, \bar{\theta}). \tag{II.1}$$

Here, the $a_i(x, y, \theta, \bar{\theta})$ are the square-matrix-valued functions. The subscript $i$ is a variable corresponding to the extra internal space which we cannot now identify. At this time, we simply regard $a_i(x, y, \theta, \bar{\theta})$ as a more fundamental field from which to construct gauge and Higgs fields, because they have only a mathematical meaning. The functions $a_i(x, y, \theta, \bar{\theta})$ do not appear in the final stage. The operator $\mathbf{d}$ in Eq.(II.1) is the generalized exterior derivative defined as follows:

$$\begin{aligned}
&\mathbf{d} = d + d_\chi + d_\theta + d_{\bar{\theta}}, &\text{(II.2)}\\
&da_i(x, y, \theta, \bar{\theta}) = \partial_\mu a_i(x, y, \theta, \bar{\theta})dx^\mu, &\text{(II.3)}\\
&d_\chi a_i(x, y, \theta, \bar{\theta}) = \partial_y a_i(x, y, \theta, \bar{\theta})\chi &\\
&\qquad\qquad = [-a_i(x, y, \theta, \bar{\theta})M(y) + M(y)a_i(x, -y, \theta, \bar{\theta})]\chi, &\text{(II.4)}\\
&d_\theta a_i(x, y, \theta, \bar{\theta}) = \partial_\theta a_i(x, y, \theta, \bar{\theta})d\theta &\text{(II.5)}\\
&d_{\bar{\theta}} a_i(x, y, \theta, \bar{\theta}) = \partial_{\bar{\theta}} a_i(x, y, \theta, \bar{\theta})d\bar{\theta}, &\text{(II.6)}
\end{aligned}$$

where the derivative $\partial_y$ is defined in Eq.(II.4), $dx^\mu$ is ordinary one-form basis, taken to be dimensionless, in Minkowski space $M_4$, and $\chi$ is the one-form basis, also assumed to be dimensionless, in the discrete space $Z_2$. The operators $d_\theta$ and $d_{\bar{\theta}}$ are also one-form bases in super-space. We have introduced the $x$-independent matrix $M(y)$, whose hermitian conjugation is given by $M(y)^\dagger = M(-y)$. The matrix $M(y)$ turns out to determine the scale and pattern of the spontaneous breakdown of the gauge symmetry. In order to find the explicit forms of gauge, Higgs and ghost fields according to Eqs. (II.1) and (II.2)∼(II.6), we need the following important algebraic rule of noncommutative geometry:

$$f(x, y, \theta, \bar{\theta})\chi = \chi f(x, -y, \theta, \bar{\theta}), \tag{II.7}$$

where $f(x, y, \theta, \bar{\theta})$ is a field such as $a_i(x, y, \theta, \bar{\theta})$, a gauge, Higgs, ghost or fermion field defined on the discrete space. It should be noticed that Eq.(II.7) does not express the relation between the matrix elements of $f(x, +, \theta, \bar{\theta})$ and $f(x, -, \theta, \bar{\theta})$, but rather insures the product between the fields expressed in differential form on the discrete space. Its justification can be easily seen in the calculation of the wedge product $\mathcal{A}(x, y, \theta, \bar{\theta})\wedge\mathcal{A}(x, y, \theta, \bar{\theta})$. Equation(II.7) realizes the non-commutativity of our algebra in the geometry on the discrete space $M_4 \times Z_2$. Inserting Eq.(II.2)∼Eq.(II.6) into Eq.(II.1) and using Eq.(II.7), $\mathcal{A}(x, y, \theta, \bar{\theta})$ is rewritten as

$$\mathcal{A}(x, y, \theta, \bar{\theta}) = A_\mu(x, y, \theta, \bar{\theta})dx^\mu + \Phi(x, y, \theta, \bar{\theta})\chi + C(x, y, \theta, \bar{\theta})d\theta + \bar{C}(x, y, \theta, \bar{\theta})d\bar{\theta}, \tag{II.8}$$

where

$$A_\mu(x, y, \theta, \bar{\theta}) = \sum_i a_i^\dagger(x, y, \theta, \bar{\theta})\partial_\mu a_i(x, y, \theta, \bar{\theta}), \tag{II.9}$$

$$\Phi(x, y, \theta, \bar{\theta}) = \sum_i a_i^\dagger(x, y, \theta, \bar{\theta})\ \left[-a_i(x, y, \theta, \bar{\theta})M(y) + M(y)a_i(x, -y, \theta, \bar{\theta})\right], \tag{II.10}$$

$$C(x, y, \theta, \bar{\theta}) = \sum_i a_i^\dagger(x, y, \theta, \bar{\theta})\partial_\theta a_i(x, y\theta, \bar{\theta}) \tag{II.11}$$

$$\bar{C}(x, y, \theta, \bar{\theta}) = \sum_i a_i^\dagger(x, y, \theta, \bar{\theta})\partial_{\bar{\theta}} a_i(x, y, \theta, \bar{\theta}). \tag{II.12}$$

Here, $A_\mu(x,y,\theta,\bar\theta)$, $\Phi(x,y,\theta,\bar\theta)$, $C(x,y,\theta,\bar\theta)$ and $\bar C(x,y,\theta,\bar\theta)$ are identified with the gauge field in the flavor symmetry, Higgs field, ghost and anti-ghost fields, respectively. In order to identify $A_\mu(x,y,\theta,\bar\theta)$ with true gauge fields, the following conditions must be imposed:

$$\sum_i a_i^\dagger(x,y,\theta,\bar\theta)a_i(x,y,\theta,\bar\theta)=1. \tag{II.13}$$

Equation(II.13) reminds us of the effective gauge field of the Berry phase [18], though the parameter space of $i$ and Minkowski space of $x^\mu$ are reversed. This might be a key to identifying the fundamental field $a_i(x,y,\theta,\bar\theta)$. Equation (II.13) is very important and we use it often below. For later convenience, we define the one-form fields as

$$\hat A(x,y,\theta,\bar\theta)=A_\mu(x,y,\theta,\bar\theta)dx^\mu, \tag{II.14}$$
$$\hat\Phi(x,y,\theta,\bar\theta)=\Phi(x,y,\theta,\bar\theta)\chi, \tag{II.15}$$
$$\hat C(x,y,\theta,\bar\theta)=C(x,y,\theta,\bar\theta)d\theta, \tag{II.16}$$
$$\hat{\bar C}(x,y,\theta,\bar\theta),=\bar C(x,y,\theta,\bar\theta)d\bar\theta. \tag{II.17}$$

Before constructing the gauge covariant field strength, we address the gauge transformation of $a_i(x,y,\theta,\bar\theta)$, which is defined as

$$a_i^g(x,y,\theta,\bar\theta)=a_i(x,y,\theta,\bar\theta)g(x,y), \tag{II.18}$$

where $g(x,y)$ is the gauge function with respect to the corresponding flavor unitary group. Then, we can find from Eqs.(II.1) and (II.18) the gauge transformation of $\mathcal{A}(x,y,\theta,\bar\theta)$ to be

$$\mathcal{A}^g(x,y,\theta,\bar\theta)=g^{-1}(x,y)\mathcal{A}(x,y,\theta,\bar\theta)g(x,y)+g^{-1}(x,y)\mathbf{d}g(x,y), \tag{II.19}$$

where, as in Eq.(II.2)∼Eq.(II.4),

$$\begin{aligned}\mathbf{d}g(x,y)&=(d+d_\chi)g(x,y)=\partial_\mu g(x,y)dx^\mu+\partial_y g(x,y)\chi\\ &=\partial_\mu g(x,y)dx^\mu+[-g(x,y)M(y)+M(y)g(x,-y)]\chi.\end{aligned} \tag{II.20}$$

Using Eqs.(II.18) and (II.19), we can find the gauge transformations of gauge, Higgs, ghost and anti-ghost fields as

$$A_\mu^g(x,y,\theta,\bar\theta)=g^{-1}(x,y)A_\mu(x,y,\theta,\bar\theta)g(x,y)+g^{-1}(x,y)\partial_\mu g(x,y), \tag{II.21}$$
$$\Phi^g(x,y,\theta,\bar\theta)=g^{-1}(x,y)\Phi(x,y,\theta,\bar\theta)g(x,-y)+g^{-1}(x,y)\partial_y g(x,y), \tag{II.22}$$
$$C^g(x,y,\theta,\bar\theta)=g^{-1}(x,y)C(x,y,\theta,\bar\theta)g(x,y), \tag{II.23}$$
$$\bar C^g(x,y,\theta,\bar\theta)=g^{-1}(x,y)\bar C(x,y,\theta,\bar\theta)g(x,y). \tag{II.24}$$

Equation(II.22) is very similar to Eq.(II.21), the gauge transformation of the genuine gauge field $A_\mu(x,y,\theta,\bar\theta)$. This strongly suggests that the Higgs field is a kind of gauge field on the discrete space $M_4\times Z_2$. From Eq.(II.20), Eq.(II.22) is rewritten as

$$\Phi^g(x,y,\theta,\bar\theta)+M(y)=g^{-1}(x,y)(\Phi(x,y,\theta,\bar\theta)+M(y))g(x,-y) \tag{II.25}$$

which makes obvious the fact that $H(x,y,\theta,\bar\theta)$ defined as

$$H(x,y,\theta,\bar\theta)=\Phi(x,y,\theta,\bar\theta)+M(y) \tag{II.26}$$

is the un-shifted Higgs field, whereas $\Phi(x,y,\theta,\bar\theta)$ denotes the shifted Higgs field with vanishing vacuum expectation value. Equations (II.23) and (II.24) demonstrate that ghost and anti-ghost fields are transformed as the field in adjoint representation.

In addition to the algebraic rules in Eqs.(II.2)∼(II.6), we add one more important rule,

$$d_\chi M(y)=M(y)M(-y)\chi, \tag{II.27}$$

which together with Eq.(II.4) yields the nilpotency of $\chi$ and, in turn, the nilpotency of the generalized exterior derivative $\mathbf{d}$ :

$$\mathbf{d}^2 f(x,y,\theta,\bar\theta)=(d^2+d_\chi^2+d_\theta^2+d_{\bar\theta}^2)f(x,y,\theta,\bar\theta)=0, \tag{II.28}$$

with the natural conditions

$$
\begin{aligned}
&dx^\mu \wedge \chi = -\chi \wedge dx^\mu, \qquad && dx^\mu \wedge d\theta = -d\theta \wedge dx^\mu, \qquad && dx^\mu \wedge d\bar{\theta} = -d\bar{\theta} \wedge dx^\mu, \\
&\chi \wedge d\theta = -d\theta \wedge \chi, && \chi \wedge d\bar{\theta} = -d\bar{\theta} \wedge \chi, && \\
&d\theta \wedge d\bar{\theta} = d\bar{\theta} \wedge d\theta, && \partial_\theta \partial_{\bar{\theta}} = -\partial_{\bar{\theta}} \partial_\theta. && \qquad \text{(II.29)}
\end{aligned}
$$

For proof of the nilpotency of $d_\chi$, see Ref. [10]. With these considerations we can construct the gauge covariant field strength:

$$
\mathcal{F}(x, y, \theta, \bar{\theta}) = \mathbf{d}\mathcal{A}(x, y, \theta, \bar{\theta}) + \mathcal{A}(x, y, \theta, \bar{\theta}) \wedge \mathcal{A}(x, y, \theta, \bar{\theta}). \tag{II.30}
$$

From Eqs.(II.19) and (II.28) we easily find the gauge transformation of $\mathcal{F}(x, y, \theta, \bar{\theta})$ as

$$
\mathcal{F}^g(x, y, \theta, \bar{\theta}) = g^{-1}(x, y)\mathcal{F}(x, y, \theta, \bar{\theta})g(x, y). \tag{II.31}
$$

Here, following Bonora and Tonin [17], we impose the horizontality condition on $\mathcal{F}(x, y, \theta, \bar{\theta})$:

$$
\mathcal{F}(x, y, \theta, \bar{\theta})|_{\theta=\bar{\theta}=0} = F(x, y), \tag{II.32}
$$

where $F(x, y)$ is the generalized field strength not accompanying the one-form bases $d\theta$ and $d\bar{\theta}$. Equation(II.32) yields the following conditions:

$$
\begin{aligned}
&d_\theta \hat{A}(x, y) + d\hat{C}(x, y) + \hat{A}(x, y) \wedge \hat{C}(x, y) + \hat{C}(x, y) \wedge \hat{A}(x, y) = 0, && \text{(II.33)} \\
&d_{\bar{\theta}} \hat{A}(x, y) + d\hat{\bar{C}}(x, y) + \hat{A}(x, y) \wedge \hat{\bar{C}}(x, y) + \hat{\bar{C}}(x, y) \wedge \hat{A}(x, y) = 0, && \text{(II.34)} \\
&d_\theta \hat{\Phi}(x, y) + d_\chi \hat{C}(x, y) + \hat{\Phi}(x, y) \wedge \hat{C}(x, y) + \hat{C}(x, y) \wedge \hat{\Phi}(x, y) = 0, && \text{(II.35)} \\
&d_{\bar{\theta}} \hat{\Phi}(x, y) + d_\chi \hat{\bar{C}}(x, y) + \hat{\Phi}(x, y) \wedge \hat{\bar{C}}(x, y) + \hat{\bar{C}}(x, y) \wedge \hat{\Phi}(x, y) = 0, && \text{(II.36)} \\
&d_\theta \hat{C}(x, y) + \hat{C}(x, y) \wedge \hat{C}(x, y) = 0, && \text{(II.37)} \\
&d_{\bar{\theta}} \hat{\bar{C}}(x, y) + \hat{\bar{C}}(x, y) \wedge \hat{\bar{C}}(x, y) = 0, && \text{(II.38)} \\
&d_{\bar{\theta}} \hat{C}(x, y) + d_\theta \hat{\bar{C}}(x, y) + \hat{C}(x, y) \wedge \hat{\bar{C}}(x, y) + \hat{\bar{C}}(x, y) \wedge \hat{C}(x, y) = 0. && \text{(II.39)}
\end{aligned}
$$

These determine the BRS/anti-BRS transformations of the respective fields. Strictly speaking, $d_\theta \hat{A}(x, y)$ in Eq.(II.33) should be written as $d_\theta \hat{A}(x, y, \theta, \bar{\theta})|_{\theta=\bar{\theta}=0}$. The notations used in Eqs.(II.34)$\sim$(II.39) follows the same convention. Nakanishi-Lautrup fields are defined as

$$
d_\theta \hat{\bar{C}}(x, y) = \hat{B}(x, y), \qquad\qquad d_{\bar{\theta}} \hat{C}(x, y) = \hat{\bar{B}}(x, y). \tag{II.40}
$$

It should be noted that the nilpotencies of $d_\theta$ and $d_{\bar{\theta}}$ are consistent with Eqs.(II.33)$\sim$(II.40). We would like to draw attention to Eqs.(II.35) and (II.36), which contain terms of the BRST/anti-BRST of the Higgs field. From Eq.(II.15)$\sim$Eq.(II.17), these two equations can be rewritten as

$$
\begin{aligned}
\partial_\theta \Phi(x, y) &= \partial_y C(x, y) + \Phi(x, y)C(x, -y) - C(x, y)\Phi(x, y), && \text{(II.41)} \\
\partial_{\bar{\theta}} \Phi(x, y) &= \partial_y \bar{C}(x, y) + \Phi(x, y)\bar{C}(x, -y) - \bar{C}(x, y)\Phi(x, y), && \text{(II.42)}
\end{aligned}
$$

which by use of the relation $H(x, y) = \Phi(x, y) + M(y)$ lead to

$$
\begin{aligned}
\partial_\theta H(x, y) &= H(x, y)C(x, -y) - C(x, y)H(x, y), && \text{(II.43)} \\
\partial_{\bar{\theta}} H(x, y) &= H(x, y)\bar{C}(x, -y) - \bar{C}(x, y)H(x, y). && \text{(II.44)}
\end{aligned}
$$

Equations (II.43) and (II.44) are the usual BRST/anti-BRST of the Higgs field.

The BRST invariant Yang-Mills-Higgs Lagrangian is obtained as

$$
\begin{aligned}
\mathcal{L}_{\mathrm{YMH}} = &- \sum_{y=\pm} \frac{1}{g_y^2} \mathrm{Tr} < F(x, y), F(x, y) > \\
&+ \sum_{y=\pm} \frac{1}{g_y^2} i\partial_\theta \partial_{\bar{\theta}} \mathrm{Tr} < \mathcal{A}(x, y, \theta, \bar{\theta}), \mathcal{A}(x, y, \theta, \bar{\theta}) > |_{\theta=\bar{\theta}=0} \\
&+ \sum_{y=\pm} \frac{1}{g_y^2} \frac{\alpha}{2} \mathrm{Tr} < \hat{B}(x, y, \theta, \bar{\theta}), \hat{B}(x, y, \theta, \bar{\theta}) > |_{\theta=\bar{\theta}=0}, \qquad \text{(II.45)}
\end{aligned}
$$

where $g_y$ is a constant related to the coupling constant of the flavor gauge field, and Tr denotes the trace over internal symmetry matrices. In order to express the Yang-Mills-Higgs Lagrangian , let us denote the explicit expressions of the field strength $F(x,y)$. The algebraic rules defined in Eqs.(II.2)∼(II.4), (II.7) and (II.13) yield

$$\begin{aligned} F(x,y) = &\frac{1}{2}F_{\mu\nu}(x,y)dx^\mu \wedge dx^\nu \\ &+D_\mu\Phi(x,y)dx^\mu \wedge \chi + V(x,y)\chi\wedge\chi, \end{aligned} \tag{II.46}$$

where

$$F_{\mu\nu}(x,y) = \partial_\mu A_\nu(x,y) - \partial_\nu A_\mu(x,y) + [A_\mu(x,y), A_\mu(x,y)], \tag{II.47}$$

$$\begin{aligned} D_\mu\Phi(x,y) = &\partial_\mu\Phi(x,y) + A_\mu(x,y)(M(y)+\Phi(x,y)) \\ &-(\Phi(x,y)+M(y))A_\mu(x,-y), \end{aligned} \tag{II.48}$$

$$V(x,y) = (\Phi(x,y)+M(y))(\Phi(x,-y)+M(-y)) - Y(x,y). \tag{II.49}$$

The function $Y(x,y)$ in Eq.(II.49) is an auxiliary field expressed as

$$Y(x,y) = \sum_i a_i^\dagger(x,y)M(y)M(-y)a_i(x,y) \tag{II.50}$$

which may or may not depend on $\Phi(x,y)$ and/or may be a constant field.

In order to obtain an explicit expression for $L_{\rm YMH}$ in Eq.(II.45) we must determine the metric structure of one-forms.

$$\begin{aligned} &< dx^\mu, dx^\nu >= g^{\mu\nu}, \quad g^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1), \\ &< \chi,\chi >= -1, \qquad < d\theta, d\theta >=< d\bar{\theta}, d\bar{\theta} >= 1. \end{aligned} \tag{II.51}$$

We note here that all other such combinations vanish. From Eqs.(II.46)∼(II.49), the first term of Eq.(II.45), that representing the gauge and Higgs bosons sector, is written as

$$\begin{aligned} \mathcal{L}^{B}_{\rm YMH} = &-\mathrm{Tr}\sum_{y=\pm}\frac{1}{2g_y^2}F^\dagger_{\mu\nu}(x,y)F^{\mu\nu}(x,y) \\ &+\mathrm{Tr}\sum_{y=\pm}\frac{1}{g_y^2}\left[D_\mu\Phi(x,y)\right]^\dagger D^\mu\Phi(x,y) \\ &-\mathrm{Tr}\sum_{y=\pm}\frac{1}{g_y^2}V^\dagger(x,y)V(x,y), \end{aligned} \tag{II.52}$$

where the third term on the right hand side of Eq.(II.52) is the potential term of the Higgs particle. The second and third terms of Eq.(II.45) give the ghost term $\mathcal{L}_{\rm GH}$ and the gauge fixing term $\mathcal{L}_{\rm GF}$, respectively. $\mathcal{L}_{\rm GH}$ is expressed as

$$\begin{aligned} \mathcal{L}_{\rm GH} = &-2i\sum_{y=\pm}\frac{1}{g_y^2}\mathrm{Tr}\partial_\mu\bar{C}(x,y)\mathcal{D}^\mu C(x,y) \\ &-i\sum_{y=\pm}\frac{1}{g_y^2}\mathrm{Tr}\left[\partial_y\bar{C}(x,-y)\mathcal{D}_yC(x,y) + \partial_y\bar{C}(x,y)\mathcal{D}_yC(x,-y)\right], \end{aligned} \tag{II.53}$$

where

$$\mathcal{D}^\mu C(x,y) = \partial^\mu C(x,y) + [A^\mu(x,y), C(x,y)], \tag{II.54}$$

$$\partial_y\bar{C}(x,y) = -\bar{C}(x,y)M(y) + M(y)\bar{C}(x,-y), \tag{II.55}$$

$$\begin{aligned} \mathcal{D}_yC(x,y) &= \partial_yC(x,y) - C(x,y)\Phi(x,y) + \Phi(x,y)C(x,-y) \\ &= -C(x,y)H(x,y) + H(x,y)C(x,-y) \end{aligned} \tag{II.56}$$

and $\mathcal{L}_{\rm GF}$ as

$$\begin{aligned} \mathcal{L}_{\rm GF} = &\frac{\alpha}{2}\sum_{y=\pm}\frac{1}{g_y^2}\mathrm{Tr}B(x,y)^2 + 2i\sum_{y=\pm}\frac{1}{g_y^2}\mathrm{Tr}\partial_\mu B(x,y)A^\mu(x,y) \\ &+i\sum_{y=\pm}\frac{1}{g_y^2}\mathrm{Tr}\left(\partial_yB(x,-y)\Phi(x,y) + \Phi(x,-y)\partial_yB(x,y)\right). \end{aligned} \tag{II.57}$$

If we note the Hermitian conjugate conditions that

$$
\begin{aligned}
&\left(\partial_y \bar{C}(x,y)\right)^\dagger = \partial_y \bar{C}(x,-y), \qquad\qquad (\mathcal{D}_y C(x,y))^\dagger = \mathcal{D}_y C(x,-y) \\
&(\partial_y B(x,y))^\dagger = -\partial_y B(x,-y)
\end{aligned}
\tag{II.58}
$$

due to $B(x,y)^\dagger = B(x,y)$, $C(x,y)^\dagger = -C(x,y)$ and $\bar{C}(x,y)^\dagger = -\bar{C}(x,y)$, we easily find the Hermiticity of Eqs.(II.53) and (II.57).

There is another method to obtain the potential term of the Higgs field. Sitarz [8] defined the new metric $g_{\alpha\beta}$ with $\alpha$ and $\beta$ running over $0,1,2,3,4$ by $g^{\alpha\beta} = \mathrm{diag}(+,-,-,-,-)$. The fifth index here represents the discrete space $Z_2$. Then, $dx^\alpha = (dx^0, dx^1, dx^2, dx^3, \chi)$ is followed. The generalized field strength $F(x,y)$ in Eq.(II.46) is written as $F(x,y) = F_{\alpha\beta}(x,y) dx^\alpha \wedge dx^\beta$, where $F_{\alpha\beta}(x,y)$ is defined in Eqs.(II.47)$\sim$(II.49). Then, it is easily found that $\mathrm{Tr}\{g^{\alpha\beta} F_{\alpha\beta}(x,y)\}$ is gauge invariant. Thus, the term

$$
|\mathrm{Tr}\{g^{\alpha\beta} F_{\alpha\beta}(x,y)\}|^2 = \{\mathrm{Tr}V(x,y)\}^\dagger \{\mathrm{Tr}V(x,y)\}
\tag{II.59}
$$

can be added to Eq.(II.52). If this term exists, the restriction between coupling constants is lost, and thus, the Higgs mass becomes a free parameter.

## III. APPLICATION TO THE SU(2)×U(1) GAUGE MODEL

In this section we apply the results of the previous section to the spontaneously broken SU(2)×U(1) gauge model. We do not deal with the Fermion sector in this article because it is not related to the main theme of this article. Let us first assign the fields on the discrete space $M_4 \times Z_2$ to the fields in the SU(2)×U(1) gauge model. For gauge fields

$$
\begin{aligned}
A_\mu(x,+) &= -\frac{i}{2}\sum_{i=1}^{3} \tau^i A^i_\mu(x) - \frac{i}{2} a \tau^0 B_\mu(x), \\
A_\mu(x,-) &= -\frac{i}{2} b B_\mu(x),
\end{aligned}
\tag{III.1}
$$

where $A^i_\mu(x)$ and $B_\mu(x)$ denote SU(2) and U(1) gauge fields, respectively. Here, $\tau^i (i=1,2,3)$ are Pauli matrices, and $\tau^0$ is the $2\times 2$ unit matrix. The condition $a - b = 1$ is required because it corresponds to the hypercharge of the Higgs field. The Higgs field is assigned as

$$
\begin{aligned}
&\Phi(x,+) = \Phi(x) = \begin{pmatrix} \phi_+(x) \\ \phi_0(x) \end{pmatrix}, \qquad\qquad M(+) = \begin{pmatrix} 0 \\ \mu \end{pmatrix}, \\
&\Phi(x,-) = \Phi^\dagger(x), \qquad\qquad M(-) = (0, \;\; \mu) = M^\dagger(+),
\end{aligned}
\tag{III.2}
$$

where $M(y)$ must be chosen to give the correct symmetry breakdown. For ghost and anti-ghost fields, which correspond with gauge fields in Eq.(III.1), we take

$$
\begin{aligned}
C(x,+) &= -\frac{i}{2}\sum_{i=1}^{3} \tau^i C^i(x) - \frac{i}{2} a \tau^0 C^0(x), \\
C(x,-) &= -\frac{i}{2} b C^0(x)
\end{aligned}
\tag{III.3}
$$

and

$$
\begin{aligned}
\bar{C}(x,+) &= -\frac{i}{2}\sum_{i=1}^{3} \tau^i \bar{C}^i(x) - \frac{i}{2} a \tau^0 \bar{C}^0(x), \\
\bar{C}(x,-) &= -\frac{i}{2} b \bar{C}^0(x).
\end{aligned}
\tag{III.4}
$$

Also for the Nakanishi-Lautrup field, we assign

$$B(x,+) = \frac{1}{2}\sum_{i=1}^{3}\tau^i B^i(x) + \frac{1}{2}a\tau^0 B^0(x),$$
$$B(x,-) = \frac{1}{2}bB^0(x) \tag{III.5}$$

and

$$\bar{B}(x,+) = \frac{1}{2}\sum_{i=1}^{3}\tau^i \bar{B}^i(x) + \frac{1}{2}a\tau^0 \bar{B}^0(x),$$
$$\bar{B}(x,-) = \frac{1}{2}b\bar{B}^0(x), \tag{III.6}$$

because $\partial_\theta \bar{C}^i = iB^i$ and $\partial_\theta \bar{C}^0 = iB^0$. We can take the gauge transformation functions as

$$g(x,+) = e^{-ia\alpha(x)}g(x),\ e^{-ia\alpha(x)} \in \mathrm{U}(1),\ g(x) \in \mathrm{SU}(2),$$
$$g(x,-) = e^{-ib\alpha(x)} \in \mathrm{U}(1). \tag{III.7}$$

After elimination of the auxiliary field $Y(x,+)$ and the rescaling of fields

$$A^i_\mu(x) \to gA^i_\mu(x), \qquad B_\mu(x) \to g'B_\mu(x), \qquad H(x) \to g_H H(x), \tag{III.8}$$

with $g = g_+,\ g^{'} = g_+g_-/\sqrt{a^2g_-^2 + b^2g_+^2/2}$ and $g_H = g_+g_-/\sqrt{g_+^2+g_-^2}$, we find the standard Yang-Mills-Higgs Lagrangian for the SU(2)×U(1) gauge theory:

$$\mathcal{L}^{B}_{\mathrm{YMH}} = -\frac{1}{4}\left[\sum_i F^i_{\mu\nu}(x)\cdot F^{i\mu\nu}(x) + B_{\mu\nu}(x)\cdot B^{\mu\nu}(x)\right]$$
$$+ [D_\mu H(x)]^\dagger\,[D^\mu H(x)] - \lambda\left[H^\dagger(x)\,H(x) - \mu^2\right]^2, \tag{III.9}$$

where

$$F^i_{\mu\nu}(x) = \partial_\mu A^i_\nu(x) - \partial_\nu A^i_\mu(x) + g\,\epsilon^{ijk}A^j_\mu(x)A^j_\nu(x), \tag{III.10}$$
$$B_{\mu\nu}(x) = \partial_\mu B_\nu(x) - \partial_\nu B_\mu(x) \tag{III.11}$$
$$D_\mu H(x) = \left[\partial_\mu - \frac{i}{2}(g\sum_i \tau^i\cdot A^i_\mu(x) + g'\tau_0 B_\mu(x))\right]H(x), \tag{III.12}$$
$$\lambda = g_+^4g_-^2/(g_+^2+g_-^2)^2\,, \qquad \mu \to g_H\mu.$$

Equation(III.9) expresses the Yang-Mills-Higgs Lagrangian of the gauge theory with the symmetry SU(2)×U(1) spontaneously broken to $\mathrm{SU(1)_{em}}$. If the Sitarz term in Eq.(II.59) is added, the coupling constant $\lambda$ is changed to $\lambda'$, removing the restriction between coupling constants, and therefore making it possible to perform the renormalization of the gauge theory as usual.

Let us now move to the ghost and gauge fixing terms expressed in Eqs.(II.53) and (II.57). For simplicity, we hereafter omit the argument $x$ in the respective fields. After applying the same rescaling of ghost and Nakanishi-Lautrup fields as that applied in Eq.(III.8)

$$B^i \to gB^i, \qquad B^0 \to g'B^0, \qquad \bar{B}^i \to g\bar{B}^i, \qquad \bar{B}^0 \to g'\bar{B}^0,$$
$$C^i \to gC^i, \qquad C^0 \to g'C^0 \qquad \bar{C}^i \to g\bar{C}^i, \qquad \bar{C}^0 \to g'\bar{C}^0, \tag{III.13}$$

we obtain the gauge fixing term $\mathcal{L}_{\mathrm{GF}}$ in Eq.(II.53) as

$$\mathcal{L}_{\mathrm{GF}} = \frac{\alpha}{2}({B_1}^2 + {B_2}^2 + {B_Z}^2 + {B_A}^2)$$
$$-B_1\left(\partial^\mu A^1_\mu - m_W\phi^1\right) - B_2\left(\partial^\mu A^2_\mu - m_W\phi^2\right)$$
$$-B_Z\left(\partial^\mu Z_\mu - m_Z\phi^4\right) - B_A\partial^\mu A_\mu, \tag{III.14}$$

where $A_\mu$ and $Z_\mu$ represent the photon and the neural weak boson, respectively, and other fields are defined as

$$B_A = B_0 \cos\theta_W + B_3 \sin\theta_W, \qquad B_Z = -B_0 \sin\theta_W + B_3 \cos\theta_W, \tag{III.15}$$

$$\phi^+ = \frac{1}{\sqrt{2}}(\phi^2 + i\phi^1), \qquad \phi^- = \frac{1}{\sqrt{2}}(\phi^2 - i\phi^1), \tag{III.16}$$

$$\phi^0 = \frac{1}{\sqrt{2}}(\phi^3 - i\phi^4), \qquad \phi^{0*} = \frac{1}{\sqrt{2}}(\phi^3 + i\phi^4) \tag{III.17}$$

with the Weinberg angle $\theta_W$. The quantities $m_W$ and $m_Z$ in Eq.(III.14) are the charged and neutral gauge boson masses, respectively. The equations of motion eliminate the Nakanishi-Lautrup fields from Eq.(III.14), yielding

$$\begin{aligned}\mathcal{L}_{\rm GF} = &-\frac{1}{2\alpha}\left(\partial^\mu A^1_\mu - m_W\phi^1\right)^2 - \frac{1}{2\alpha}\left(\partial^\mu A^2_\mu - m_W\phi^2\right)^2 \\ &-\frac{1}{2\alpha}\left(\partial^\mu Z_\mu - m_Z\phi^4\right)^2 - \frac{1}{2\alpha}\left(\partial^\mu A_\mu\right)^2 .\end{aligned} \tag{III.18}$$

Eq.(III.18) enables us to obtain the gauge fixed Lagrangian with the 't Hooft-Feynman gauge [19]. This result is the same as that reported in Ref. [16].

With the notations used in Eq.(III.18), we obtain the explicit expression of the ghost terms in Eq.(II.53) as follows:

$$\begin{aligned}\mathcal{L}_{\rm GH} = &\, i\partial^\mu \bar{C}^k \mathcal{D}_\mu C^k + i\partial^\mu \bar{C}^0 \partial_\mu C^0 \\ &+\frac{i}{2} m_W \left\{ g(\bar{C}^1C^1 + \bar{C}^2C^2)(\phi^3 + v) + g(\bar{C}^2C^1 - \bar{C}^1C^2)\phi^4 \right. \\ &\qquad \left. + (gC^3 + g'C^0)(\bar{C}^1\phi^1 - \bar{C}^2\phi^2) \right\} \\ &+\frac{i}{2} m_Z \left\{ -gC_Z(C^1\phi^1 - C^2\phi^2) + \sqrt{g^2 + g'^2}\,\bar{C}_Z C_Z(\phi^3 + v) \right\},\end{aligned} \tag{III.19}$$

where $v = \sqrt{2}\mu$, $\mathcal{D}_\mu C^k = \partial_\mu C^k + \epsilon^{klm} A^l_\mu C^m$ and

$$C_Z = -C_0 \sin\theta_W + C_3 \cos\theta_W, \qquad \bar{C}_Z = -\bar{C}_0 \sin\theta_W + \bar{C}_3 \cos\theta_W. \tag{III.20}$$

The new interaction terms between ghost and Higgs fields appear here. This is a natural result because the Higgs field is a member of the generalized connection in NCG on the discrete space just as the gauge fields $A_\mu, W^\pm_\mu$ and $Z_\mu$.

## IV. CONCLUDING REMARKS

The BRST invariant Lagrangian of the spontaneously broken gauge theory is presented in our scheme by introducing the Grassmann numbers $\theta$ and $\bar{\theta}$ as the arguments in super space in addition to $x_\mu$ in $M_4$ and $y$ in $Z_2$. The horizontality condition prescribes the BRS transformations of the respective fields, including the Higgs field. By use of the generalized gauge field $\mathcal{A}(x, y, \theta, \bar{\theta})$ and the Nakanishi-Lautrup field $\bar{B}(x, y, \theta, \bar{\theta})$ as in Eq.(II.45), the gauge fixing and ghost terms appear in the Lagrangian, yielding the t'Hooft-Feynman gauge as the gauge fixing condition and extra interactions between ghost and Higgs fields.

If we include in the Yang-Mills-Higgs Lagrangian the quartic term of the Higgs field due to Sitarz [8] in Eq.(II.59), it attains a form which is free from any restriction on coupling constants, making it possible to quantize the BRST invariant Lagrangian in the same way as the ordinary Lagrangian. Generally speaking, the NCG approach to clarify the Higgs mechanism provides the estimations of the Weinberg angle and the mass relation between the Higgs and gauge bosons or top quark. These estimations may be in a sense inconsistent with the quantization of the theory. However, some of these relations [15], [21] are too attractive to be discarded. Alvarez, Gracia-Bondía and Martín [20] in fact calculated such NCG constraints using the renormalization group analysis and obtained interesting relations, including $m_{top} = 2m_W$ and $m_H = 3.14m_W$, where the NCG constraints are assumed to hold at some renormalization point. In order to perform such analyses, a renormalizable Lagrangian including the gauge fixing and ghost terms must exist. In this sense, the work of Lee, Hwang and Ne'eman [16] is very important for such renormalization analysis. The present work also shows how we can obtain the renormalizable Lagrangian. In a previous work [21], we found the relations $\sin^2\theta_{_W} = 3/8$ and $m_{_H} = \sqrt{2}m_{_W}$, assumed to be satisfied at the energy of the GUT scale. In addition, we obtained the mass relation $m_H = \frac{4}{\sqrt{3}} m_W \sin\theta_W$ with the assumption that the coupling constant of the Higgs quartic term is not affected by the Sitarz term. It might be expected in the NCG scheme that the quadratic divergent term of the Higgs self-energy disappears because of gauge invariance, if the Higgs field would be the genuine

gauge field on the discrete space. If this were true, the mass relation $m_H = \frac{4}{\sqrt{3}} m_W \sin\theta_W$ might hold without any correction in the same way as $m_W = m_Z \cos\theta_W$. In any case, it is very interesting to investigate the renormalization analysis of these results according to the BRST invariant Lagrangian presented in this article. This will be done in a future work.

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